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MULTI-STAGE SYMPLECTIC SCHEMES OF TWO KINDS OF HAMILTONIAN SYSTEMS FOR WAVE EQUATIONS*

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ABSTRACT

Multi-stage schemes for wave equation are constructed. Their stability conditions are discussed.

INTRODUCTION

Recently Hamiltonian formalism plays a fundamental role in mathematical physics, classical mechanics, optics, quantum mechanics, hydrodynamics of a perfect fluid, plasma physics, and accelerator physics.

Hamiltonian formalism has the important property of being area-preserving (symplectic) i.e. the sum of the areas of canonical variable pairs, projected on any two-dimensional surface in phase space, is time invariant. In numerically solving these equations, one hopes that the approximating equations will hold this property.

In [1] Feng Kang proposed a program of systematic study of symplectic difference schemes for Hamiltonian equations from the viewpoint of symplectic geometry, and developed, with his colleagues, a systematical method–generating function method–to construct such schemes [2-4].These schemes preserve many properties of the system.

In [5] [10] [11], Some explicit symplectic schemes for separable Hamiltonian systems are obtained. In this paper, we apply them to wave equation. Multi-stage schemes for wave equation are constructed. Their stability conditions are discussed. The stability range of schemes constracts as the order of accuracy rises in spacial direction. Some schemes are stable even though the Courant number exceeds 2.

§1. THE CONSTRUCTION OF THE MULTI-STAGE SYMPLECTIC SCHEME

First we consider a finite dimension separable Hamiltonian system with Hamiltonian $H(p,q) = U(p) + V(q)$. The equation of motion is

$$\frac{d}{dt}\begin{pmatrix} p \\ q \end{pmatrix} = J^{-1}\begin{pmatrix} U_p \\ V_q \end{pmatrix} = \begin{pmatrix} f(q) \\ g(p) \end{pmatrix}$$

where $J^{-1} = \begin{pmatrix} & -I \\ I & \end{pmatrix}$, I is the $n \times n$ identity matrix.

Several explicit multi-stage symplectic schemes are obtained in [5] [10] [11], we write them in the following:

1-satge method of order 1

$$\text{(s1)} \qquad p^{k+1} = p^k + hc_1 f(q^k), \quad q^{k+1} = q^k + hd_1 g(p^{k+1}),$$

where $c_1 = d_1 = 1, \quad h = \Delta t$.

2-stage method of order 2

$$\text{(s2)} \qquad \begin{aligned} p_1 &= p^k + hc_1 f(q^k), \quad q_1 = q^k + hd_1 g(p_1), \\ p^{k+1} &= p_1 + hc_2 f(q_1), \quad q^{k+1} = q_1 + hd_2 g(p^{k+1}), \end{aligned}$$

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where $c_1 = 0, c_2 = 1, d_1 = d_2 = \frac{1}{2}$, or $d_1 = 1, d_2 = 0, c_1 = c_2 = \frac{1}{2}$

3-stage method of order 3

$$\text{(s3)}\qquad \begin{aligned} p_1 &= p^k + hc_1 f(q^k), \quad & q_1 &= q^k + hd_1 g(p_1),\\ p_2 &= p_1 + hc_2 f(q_1), \quad & q_2 &= q_1 + hd_2 g(p_2),\\ p^{k+1} &= p_2 + hc_3 f(q_2), \quad & q^{k+1} &= q_2 + hd_3 g(p^{k+1}), \end{aligned}$$

where

$$c_1 = \frac{7}{24}, \quad c_2 = \frac{3}{4}, \quad c_3 = -\frac{1}{24},$$
$$d_1 = \frac{2}{3}, \quad d_2 = -\frac{2}{3}, \quad d_3 = 1,$$

or

$$c_1 = 1, \quad c_2 = -\frac{2}{3}, \quad c_3 = \frac{2}{3},$$
$$d_1 = -\frac{1}{24}, \quad d_2 = \frac{3}{4}, \quad d_3 = \frac{7}{24},$$

4-stage method of 4-th order of accuracy

$$\text{(s4)}\qquad \begin{aligned} p_1 &= p^k + hc_1 f(q^k), \quad & q_1 &= q^k + hd_1 g(p_1),\\ p_2 &= p_1 + hc_2 f(q_1), \quad & q_2 &= q_1 + hd_2 g(p_2),\\ p_3 &= p_2 + hc_3 f(q_2), \quad & q_3 &= q_2 + hd_3 g(p_3),\\ p^{k+1} &= p_3 + hc_4 f(q_3), \quad & q^{k+1} &= q_3 + hd_4 g(p^{k+1}), \end{aligned}$$

$$c_1 = 0, \quad c_2 = c_4 = \frac{1}{3}(2+\alpha), \quad c_3 = -\frac{1}{3}(1+2\alpha),$$
$$d_1 = d_4 = \frac{1}{6}(2+\alpha), \quad d_2 = d_3 = \frac{1}{6}(1-\alpha),$$

where $\alpha = \sqrt[3]{2} + \sqrt[3]{1/2}$.

or

$$c_1 = \frac{1}{6}(2+\alpha), \quad c_2 = c_3 = \frac{1}{6}(1-\alpha), \quad c_4 = \frac{1}{6}(2+\alpha),$$
$$d_1 = \frac{1}{3}(2+\alpha), \quad d_2 = -\frac{1}{3}(1+2\alpha), \quad d_3 = \frac{1}{3}(2+\alpha), \quad d_4 = 0.$$

Now, we consider the following wave equation:

$$\text{(1.1)}\qquad w_{tt} = w_{xx}$$

with an initial condition. The boundary condition, if any, will be periodic.

We have two forms of Hamiltonian system for Equation (1.1). One of them is classical [1-3]; its Hamiltonian functional is taken as

$$H(u,v) = \frac{1}{2}\int (v^2 + u_x^2)\,dx$$

where $u = w, v = w_t$. This Hamiltonian system can be written as

$$\text{(1.2)}\qquad \frac{dz}{dt} = J^{-1} H_z,$$

where

$$z = \begin{bmatrix} v \\ u \end{bmatrix}, \quad J^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad H_z = \begin{bmatrix} \dfrac{\delta H}{\delta v} \\ \dfrac{\delta H}{\delta u} \end{bmatrix} = \begin{bmatrix} v \\ -u_{xx} \end{bmatrix}.$$

Another is

$$\frac{dz}{dt} = \mathcal{D}\frac{\delta H}{\delta z}, \qquad H(v,u) = \frac{1}{2}\int (v^2 + u^2)\, dx, \tag{1.3}$$

where $v = w_t, u = w_x$, and

$$\mathcal{D} = \begin{bmatrix} 0 & \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial x} & 0 \end{bmatrix}$$

is obviously a skew-adjoint operator.

(1.2) can be rewritten in the form

$$\frac{dz}{dt} = J^{-1}Az, \tag{1.2'}$$

where

$$J^{-1}A = \begin{bmatrix} 0 & \Delta \\ 1 & 0 \end{bmatrix}, \tag{1.4}$$

Δ being the central difference operator for $\dfrac{\partial^2}{\partial x^2}$.

Let Δ_2 be the 2-nd order central difference operator for $\dfrac{\partial^2}{\partial x^2}$:

$$\Delta_2 u_m = \frac{u(m+1) - 2u(m) + u(m-1)}{\Delta x^2}. \tag{1.5}$$

Let Δ_4 be the 4-th order central difference operator for $\dfrac{\partial^2}{\partial x^2}$:

$$\Delta_4 u_m = \frac{-u(m+2) + 16u(m+1) - 30u(m) + 16u(m-1) - u(m-2)}{12\Delta x^2}. \tag{1.6}$$

Denoting $U = [u(1), u(2), \cdots, u(n)]'$, $V = [v(1), v(2), \cdots, v(n)]'$, equation (1.2′) becomes

$$\frac{d}{dt}\begin{bmatrix} V \\ U \end{bmatrix} = \begin{bmatrix} & M \\ I & \end{bmatrix}\begin{bmatrix} V \\ U \end{bmatrix} = \begin{bmatrix} MU \\ V \end{bmatrix}. \tag{1.7}$$

M is an $n \times n$ matrix, which is either of the following:

$$M_1 : \frac{1}{\Delta x^2}\begin{bmatrix} -2 & 1 & 0 & \cdots & \cdots & 1 \\ 1 & -2 & 1 & \cdots & \cdots & 0 \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ 0 & 0 & \cdots & \cdots & -2 & 1 \\ 1 & 0 & \cdots & \cdots & 1 & -2 \end{bmatrix}, \tag{1.8}$$

$$M_2 : \frac{1}{12\Delta x^2}\begin{bmatrix} -30 & 16 & -1 & 0 & 0 & \cdots & \cdots & 0 & -1 & 16 \\ 16 & -30 & 16 & -1 & 0 & \cdots & \cdots & 0 & 0 & -1 \\ -1 & 16 & -30 & 16 & -1 & \cdots & \cdots & 0 & 0 & 0 \\ & \ddots & \ddots & \ddots & \ddots & \ddots & & & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & & & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & & & & \ddots & \ddots & \ddots & \ddots & \ddots \\ -1 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 16 & -30 & 16 \\ 16 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & -1 & 16 & -30 \end{bmatrix}. \tag{1.9}$$

according to (1.5), (1.6) and the periodic boundary condition.

For the second kind of Hamiltonian system (1.3). We have a form similar to (1.2′), but

$$K^{-1}A = \begin{bmatrix} 0 & \nabla \\ \nabla & 0 \end{bmatrix}$$

where K is a nonsingular skew-adjoint operator, and ∇ is a central difference operator for $\frac{\partial}{\partial x}$.

Let ∇_2 be the central difference operator of second order for $\frac{\partial}{\partial x}$:

$$\nabla_2 u_m = \frac{u(m+1) - u(m-1)}{2\Delta x}. \tag{1.10}$$

Let ∇_4 be the central difference operator of fourth order:

$$\nabla_4 u_m = \frac{-u(m+2) + 8u(m+1) - 8u(m-1) + u(m-2)}{12\Delta x}. \tag{1.11}$$

We rewrite (1.3) in the following form

$$\frac{d}{dt}\begin{bmatrix} V \\ U \end{bmatrix} = \begin{bmatrix} & M \\ M & \end{bmatrix}\begin{bmatrix} V \\ U \end{bmatrix} = \begin{bmatrix} MU \\ MV \end{bmatrix}. \tag{1.12}$$

Here M is an $n \times n$ matrix. According to (1.10), (1.11), M may be put into two forms:

$$M_3 : \frac{1}{2\Delta x}\begin{bmatrix} 0 & 1 & 0 & \cdots & \cdots & -1 \\ -1 & 0 & 1 & \cdots & \cdots & 0 \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ 0 & 0 & \cdots & \cdots & 0 & 1 \\ 1 & 0 & \cdots & \cdots & -1 & 0 \end{bmatrix}. \tag{1.13}$$

$$M_4 : \frac{1}{12\Delta x}\begin{bmatrix} 0 & 8 & -1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & -8 \\ -8 & 0 & 8 & -1 & \cdots & \cdots & \cdots & \cdots & 0 & 1 \\ 1 & -8 & 0 & 8 & -1 & & & & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & & & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & & & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & & & & \ddots & \ddots & \ddots & \ddots & \ddots \\ -1 & 0 & 0 & \cdots & \cdots & \cdots & 1 & -8 & 0 & 8 \\ 8 & -1 & 0 & 0 & \cdots & \cdots & \cdots & 1 & -8 & 0 \end{bmatrix}. \tag{1.14}$$

Applying schemes (s1–s4) to equation (1.7), we obtain symplectic difference schemes for wave equation (1.2).

$$V^{k+1} = V^k + hMU^k \quad U^{k+1} = U^k + hV^{k+1}, \tag{1.15}$$

$$\begin{cases} V_1 = V^k \quad U_1 = U^k + \frac{1}{2}hV_1, \\ V^{k+1} = V_1 + hMU_1 \quad U^{k+1} = U_1 + \frac{1}{2}hV^{k+1}, \end{cases} \tag{1.16}$$

$$\begin{cases} V_1 = V^k + \frac{7}{24}hMU^k \quad U_1 = U^k + \frac{2}{3}hV_1, \\ V_2 = V_1 + \frac{3}{4}hMU_1 \quad U_2 = U_1 - \frac{2}{3}hV_2, \\ V^{k+1} = V_2 - \frac{1}{24}hMU_2 \quad U^{k+1} = U_2 + hV^{k+1}, \end{cases} \tag{1.17}$$

$$\begin{cases} V_1 = V^k \qquad U_1 = U^k + \frac{1}{6}(2+\alpha)hV_1, \\ V_2 = V_1 + \frac{1}{3}(2+\alpha)hMU_1 \quad U_2 = U_1 + \frac{1}{6}(1-\alpha)hV_2, \\ V_3 = V_2 - \frac{1}{3}(1+2\alpha)hMU_2 \quad U_3 = U_2 + \frac{1}{6}(1-\alpha)hV_3, \\ V^{k+1} = V_3 + \frac{1}{3}(2+\alpha)hMU_3 \quad U^{k+1} = U_2 + \frac{1}{6}(1-\alpha)hV^{k+1}. \end{cases} \tag{1.18}$$

If we take $M = M_1$ in schemes (1.15)–(1.18). (1.15)–(1.18) are, respectively; of order $o(\Delta t + \Delta x^2)$, $o(\Delta t^2 + \Delta x^2)$, $o(\Delta t^3 + \Delta x^2)$, $o(\Delta t^4 + \Delta x^2)$. We denote these schemes $SFW(1.2)$, $SFW(2.2)$, $SFW(3.2)$, $SFW(4.2)$. If $M = M_2$ in schemes (1.15)–(1.18). (1.15)–(1.18) are, respectively, of order $o(\Delta t + \Delta x^4)$, $o(\Delta t^2 + \Delta x^4)$, $o(\Delta t^3 + \Delta x^4)$, $o(\Delta t^4 + \Delta x^4)$. We denote these schemes $SFW(1.4)$, $SFW(2.4)$, $SFW(3.4)$, $SFW(4.4)$.

2. Stability of the Schemes

In order to study the stability of difference schemes we first quote some related criterions obtained by Miller [5].

To a polynomial of degree n with real or complex coefficients

$$f(z) = a_0 + a_1 z + \cdots + a_n z, \qquad a_0 \cdot a_n \neq 0, \tag{2.1}$$

we associate another polynomial f^* defined by

$$f^*(z) = \bar{a}_n + \bar{a}_{n-1} z + \cdots + \bar{a}_0 z^n,$$

where $\bar{a}_i$ is the complex conjugate to a_i.

If

$$f^*(0) f(z) \equiv f(0) f^*(z), \tag{2.2}$$

then the polynomial $f(z)$ has all its zeros either in the unit disc or on the unit circle iff $f'(z)$ only has zeros with modulus smaller than or equal to unity.

Corollary 1:. *Polynomial $f(z) = z^2 + 2bz + 1$, $b \in R$, has all its zeros in the unit disc or on the unit circle iff $|b| \leq 1$.*

To analyse the stability of the schemes, we will consider the eigenvalues of amplification matrices of schemes.

LEMMA 1. *Let A, B, C, D be $n \times n$ matrices. if $AC = CA$. then*

$$\det\begin{vmatrix}\lambda I - A & -B \\ -C & \lambda I - D\end{vmatrix} = \det|(\lambda I - A)(\lambda I - D) - CB|$$

$$= \det|\lambda^2 I - \lambda(A + D) + AD - CB|, \tag{2.3}$$

especially, if there exists an invertible matrix P such that

$$P^{-1}AP = \begin{bmatrix}a_1 & & \\ & \ddots & \\ & & a_n\end{bmatrix}, \qquad P^{-1}BP = \begin{bmatrix}b_1 & & \\ & \ddots & \\ & & b_n\end{bmatrix},$$

$$P^{-1}CP = \begin{bmatrix}c_1 & & \\ & \ddots & \\ & & c_n\end{bmatrix}, \qquad P^{-1}DP = \begin{bmatrix}d_1 & & \\ & \ddots & \\ & & d_n\end{bmatrix},$$

then

$$\det|\lambda^2 I - \lambda(A + B) + AD - CB| = \prod_{i=1}^{n}(\lambda^2 - (a_i + d_i)\lambda + a_i d_i - c_i b_i). \tag{2.4}$$

LEMMA 2. *The eigenvalues of matrices M_1, M_2, M_3, M_4 are, respectively,*

$$\begin{aligned} &\lambda_k^{(1)} = -\frac{4}{\Delta x^2}\sin^2\frac{k\pi}{2N}, \quad \lambda_k^{(2)} = -\frac{4}{\Delta x^2}\sin^2\frac{\pi k}{2N} - \frac{4}{3\Delta x^2}\sin^4\frac{\pi k}{2N}, \\ &\lambda_k^{(3)} = \frac{i}{\Delta x}\sin\frac{\pi k}{2N}, \qquad \lambda_k^{(4)} = \frac{i}{\Delta x}\left(\frac{-1}{6}\sin 2\frac{\pi k}{2N} + \frac{8}{6}\sin\frac{\pi k}{2N}\right), \end{aligned} \qquad k = 0, 1, \cdots, 2N-1. \tag{2.5}$$

1°. The stability of scheme (1.15).

Scheme (1.15) can be rewritten in the following form

$$\begin{pmatrix} I & 0 \\ -hI & I\end{pmatrix}\begin{pmatrix}V^{k+1} \\ U^{k+1}\end{pmatrix} = \begin{pmatrix} I & hM \\ 0 & I\end{pmatrix}\begin{pmatrix}V^{k} \\ U^{k}\end{pmatrix},$$

$$\begin{pmatrix}V^{k+1} \\ U^{k+1}\end{pmatrix} = \begin{pmatrix} I & hM \\ hI & I + h^2 M\end{pmatrix}\begin{pmatrix}V^{k} \\ U^{k}\end{pmatrix}. \tag{2.6}$$

The characteristic equation of the amplification matrix $\begin{pmatrix} I & hM \\ hI & I + h^2 M\end{pmatrix}$ is

$$\prod_{k=1}^{n}(\lambda^2 - (2 + h^2\lambda_k)\lambda + 1) \tag{2.7}$$

where λ_k $(k = 1, \cdots, n)$ are proper values of matrix M. If we take $M = M_1$, $\lambda_k = -\frac{4}{\Delta x^2}\sin^2\frac{k\pi}{2N}$. From corollary 1, a stability condition of scheme SFW(1.2) is

$$|2 + h^2(-\frac{4}{\Delta x^2}\sin^2\frac{k\pi}{2N})| \le 2, \tag{2.8}$$

i.e.

$$|\frac{h}{\Delta x}| \le 1. \tag{2.9}$$

If $M = M_2$, $\lambda_k = -\frac{4}{\Delta x^2}\sin^2\frac{\pi k}{2N} - \frac{4}{3\Delta x^2}\sin^4\frac{\pi k}{2N}$. A stability condition of scheme SFW(1.4) is

$$|2 + h^2(-\frac{4}{\Delta x^2}\sin^2\frac{\pi k}{2N} - \frac{4}{3\Delta x^2}\sin^4\frac{\pi k}{2N})| \le 2 \tag{2.10}$$

i.e.

$$|\frac{h}{\Delta x}| \leq \sqrt{\frac{3}{4}} = 0.86602. \tag{2.11}$$

2°. The stability of the scheme (1.16).

We express (1.16) in the form:

$$\begin{pmatrix} I & 0 \\ -\frac{h}{2}I & I \end{pmatrix} \begin{pmatrix} V_1 \\ U_1 \end{pmatrix} = \begin{pmatrix} V^k \\ U^k \end{pmatrix},$$
$$\begin{pmatrix} I & 0 \\ -\frac{h}{2}I & I \end{pmatrix} \begin{pmatrix} V^{k+1} \\ U^{k+1} \end{pmatrix} = \begin{pmatrix} I & hM \\ 0 & I \end{pmatrix} \begin{pmatrix} V_1 \\ U_1 \end{pmatrix}.$$

The amplification matrix of the scheme is

$$\begin{pmatrix} I + \frac{h^2}{2}M & hM \\ hI + \frac{h^3}{4}M & I + \frac{h^2}{2}M \end{pmatrix}. \tag{2.12}$$

Its characteristic equation is

$$\prod_{k=1}^{n} (\lambda^2 - (2 + h^2\lambda_k)\lambda + 1), \tag{2.13}$$

which is the same as (2.7). Thus scheme (1.16) and scheme (1.15) are of the same stability condition.

3° The stability of the scheme (1.17).

The amplification matrix of the scheme is

$$\begin{pmatrix} I + \frac{1}{2}h^2M + \frac{1}{72}h^4M^2 & hM + \frac{1}{6}h^3M^2 + \frac{1}{48} \times \frac{7}{36}h^5M^3 \\ hI + \frac{1}{6}h^3M + \frac{1}{72}h^5M^2 & I + \frac{1}{2}h^2M + \frac{5}{72}h^4M^2 + \frac{1}{48} \times \frac{7}{36}h^6M^3 \end{pmatrix}. \tag{2.14}$$

Its characteristic equation is

$$\begin{aligned} &\prod_{k=1}^{n} (\lambda^2 - (1 + \frac{1}{2}h^2\lambda_k + \frac{1}{72}h^4{\lambda_k}^2 + 1 + \frac{1}{2}h^2\lambda_k + \frac{5}{72}h^4{\lambda_k}^2 + \frac{1}{48} \times \frac{7}{36}h^6{\lambda_k}^3)\lambda + 1) \\ &= \prod_{k=1}^{n} (\lambda^2 - (2 + h^2\lambda_k + \frac{1}{12}h^4{\lambda_k}^2 + \frac{1}{48} \times \frac{7}{36}h^6{\lambda_k}^3)\lambda + 1). \end{aligned} \tag{2.15}$$

A stability condition of scheme (1.17) is

$$|2 + h^2\lambda_k + \frac{1}{12}h^4{\lambda_k}^2 + \frac{1}{48} \times \frac{7}{36}h^6{\lambda_k}^3| \leq 2.$$

$$-6.28746033 \leq h^2\lambda_k \leq 0. \tag{2.16}$$

Let $M = M_1$, $\lambda_k = \lambda_k^{(1)} = -\frac{4}{\Delta x^2}\sin^2\frac{k\pi}{2N}$, The stability condition of scheme SFW(3.2) is

$$|\frac{h}{\Delta x}| \leq 1.253740.$$

Let $M = M_2$, $\lambda_k = \lambda_k^{(2)} = -\frac{4}{\Delta x^2}\sin^2\frac{\pi k}{2N} - \frac{4}{3\Delta x^2}\sin^4\frac{\pi k}{2N}$. The stability condition of scheme SFW(3.4) is

$$|\frac{h}{\Delta x}| \le 1.08577.$$

4°. The stability of scheme (1.18).

The amplification matrix of the scheme is

$$\begin{pmatrix} A_{11} & A_{22} \\ A_{21} & A_{22} \end{pmatrix},$$

where

$$\begin{aligned}
A_{11} &= I + \frac{1}{2}h^2M + \frac{1}{24}h^4M^2 - \frac{1}{3^3}\times\frac{1}{6^3}(1-\alpha)^2(2+\alpha)^3(1+2\alpha)h^6M^3,\\
A_{12} &= hM + \frac{1}{6}h^3M^2 + (\frac{1}{72}(2+\alpha) - \frac{1}{108}(2+\alpha)^2)h^5M^3,\\
A_{21} &= hI + \frac{1}{6}h^3M + (\frac{1}{144}(2+\alpha) - \frac{1}{6^3\times 3^2}(1+2\alpha)(2+\alpha)^2(1-\alpha)^2h^5M^2,\\
&+ \frac{1}{6^4}\times\frac{(-1)}{3^3}(1-2\alpha)^2(2+\alpha)^4(1+2\alpha)h^7M^3,\\
A_{22} &= I + \frac{1}{2}h^2M + \frac{1}{24}h^4M^2 + (\frac{1}{72}(2+\alpha) - \frac{1}{108}(2+\alpha)^2)\frac{1}{6}(2+\alpha)h^6M^3,
\end{aligned} \tag{2.17}$$

where $\alpha = \sqrt[3]{2} + \sqrt[3]{1/2}$. Its characteristic equation is

$$\begin{aligned}
&\prod_{k=1}^{n}(\lambda^2 - (2 + h^2\lambda_k + \frac{1}{12}h^4\lambda_k{}^2 + ((\frac{1}{72}(2+\alpha) - \frac{1}{108}(2+\alpha)^2)\frac{1}{6}(2+\alpha)\\
&\quad - \frac{1}{3^3}\times\frac{1}{6^3}(1-\alpha)^2(2+\alpha)^3(1+2\alpha))h^6\lambda_k{}^3)\lambda + 1).
\end{aligned} \tag{2.18}$$

A stability condition of scheme (1.18) is

$$\begin{aligned}
&|(2 + h^2\lambda_k + \frac{1}{12}h^4\lambda_k{}^2 + ((\frac{1}{72}(2+\alpha) - \frac{1}{108}(2+\alpha)^2)\frac{1}{6}(2+\alpha)\\
&\quad - \frac{1}{3^3}\times\frac{1}{6^3}(1-\alpha)^2(2+\alpha)^3(1+2\alpha))h^6\lambda_k{}^3)| \le 2,
\end{aligned} \tag{2.19}$$

$$-2.47559738 \le h^2\lambda_k \le 0.$$

Let $M = M_1$, $\lambda_k = -\frac{4}{\Delta x^2}\sin^2\frac{k\pi}{2N}$. The stability condition of scheme SFW(4.2) is

$$|\frac{h}{\Delta x}| \le 0.78670151. \tag{2.20}$$

Let $M = M_2$, $\lambda_k = -\frac{4}{\Delta x^2}\sin^2\frac{\pi k}{2N} - \frac{4}{3\Delta x^2}\sin^4\frac{\pi k}{2N}$. The stability condition of scheme SFW(4.4) is

$$|\frac{h}{\Delta x}| \le 0.681303536. \tag{2.21}$$

Similarly, applying scheme (s1–s4) to equation (1.12),we obtain symplectic difference schemes for equation (1.3)*

*Here, symplectic means "general symplectic" i.e, the Jacobi matrix D of its transitional transformation satisfy $D'KD = K$, K is a nonsingular skew-symmetric matrix

$$V^{k+1} = V^k + hMU^k \quad U^{k+1} = U^k + hMV^{k+1} \tag{2.22}$$

$$\begin{cases} V_1 = V^k & U_1 = U^k + \frac{1}{2}hMV_1 \\ V^{k+1} = V_1 + hMU_1 & U^{k+1} = U_1 + \frac{1}{2}hMV^{k+1} \end{cases} \tag{2.23}$$

$$\begin{cases} V_1 = V^k + \frac{7}{24}hMU^k & U_1 = U^k + \frac{2}{3}hMV_1 \\ V_2 = V_1 + \frac{3}{4}hMU_1 & U_2 = U_1 - \frac{2}{3}hMV_2 \\ V^{k+1} = V_2 - \frac{1}{24}hMU_2 & U^{k+1} = U_2 + hMV^{k+1} \end{cases} \tag{2.24}$$

$$\begin{cases} V_1 = V^k & U_1 = U^k + \frac{1}{6}(2+\alpha)hMV_1 \\ V_2 = V_1 + \frac{1}{3}(2+\alpha)hMU_1 & U_2 = U_1 + \frac{1}{6}(1-\alpha)hMV_2 \\ V_3 = V_2 - \frac{1}{3}(1+2\alpha)hMU_2 & U_3 = U_2 + \frac{1}{6}(1-\alpha)hMV_3 \\ V^{k+1} = V_3 + \frac{1}{3}(2+\alpha)hMU_3 & U^{k+1} = U_2 + \frac{1}{6}(1-\alpha)hMV^{k+1} \end{cases} \tag{2.25}$$

If we take $M = M_3$ in scheme (2.22)–(2.25), (2.22)–(2.25) are, respectively; of order $o(\Delta t + \Delta x^2)$, $o(\Delta t^2 + \Delta x^2)$, $o(\Delta t^3 + \Delta x^2)$, $o(\Delta t^4 + \Delta x^2)$. We denote these schemes $KTW(1.2)$, $KTW(2.2)$, $KTW(3.2)$, $KTW(4.2)$. If $M = M_4$ in scheme (2,22)–(2.25). (2.22)–(2.25) are, respectively, of order $o(\Delta t + \Delta x^4)$, $o(\Delta t^2 + \Delta x^4)$, $o(\Delta t^3 + \Delta x^4)$, $o(\Delta t^4 + \Delta x^4)$. We denote these schemes $KTW(1.4)$, $KTW(2.4)$, $KTW(3.4)$, $KTW(4.4)$

We sum our results in the following tables. It is easy to find that the stability range of schemes contracts as the order of accuracy rises in spacial direction. Especially, the Courant number of some schemes exceeds 1; for some scheme it even reaches 2.50748.

Table 1

Scheme	order of approx.	stability condition
$SFW(1.2)$	$o(\Delta t + \Delta x^2)$	1
$SFW(2.2)$	$o(\Delta t^2 + \Delta x^2)$	1
$SFW(3.2)$	$o(\Delta t^3 + \Delta x^2)$	1.25374
$SFW(4.2)$	$o(\Delta t^4 + \Delta x^2)$	0.78670
$SFW(1.4)$	$o(\Delta t + \Delta x^4)$	0.86602
$SFW(2.4)$	$o(\Delta t^2 + \Delta x^4)$	0.86602
$SFW(3.4)$	$o(\Delta t^3 + \Delta x^4)$	1.08577
$SFW(4.4)$	$o(\Delta t^4 + \Delta x^4)$	0.68130

Table 2

Scheme	order of approx.	stability condition
$KTW(1.2)$	$o(\Delta t + \Delta x^2)$	2
$KTW(2.2)$	$o(\Delta t^2 + \Delta x^2)$	2
$KTW(3.2)$	$o(\Delta t^3 + \Delta x^2)$	2.50748
$KTW(4.2)$	$o(\Delta t^4 + \Delta x^2)$	1.573403
$KTW(1.4)$	$o(\Delta t + \Delta x^4)$	1.457513
$KTW(2.4)$	$o(\Delta t^2 + \Delta x^4)$	1.457513
$KTW(3.4)$	$o(\Delta t^3 + \Delta x^4)$	1.827343
$KTW(4.4)$	$o(\Delta t^4 + \Delta x^4)$	1.1463570

§3. MULTI-STAGE IMPLICIT SYMPLECTIC SCHEME

For general Hamiltonian system

$$\frac{dz}{dt} = J^{-1}H_z \tag{3.1}$$

some implicit symplectic schemes are obtained in [1]. We apply them to equations (1.17) and (1.12). Several implicit symplectic schemes for equations (1.2) and (1.3) are given. Although they are implicit,it is not difficult to implement.

We write these schemes in the following:

$$\begin{cases} z^{k+1} = 2Y_1 - z^k \\ Y_1 = z^k + \frac{1}{2}hSY_1 \end{cases} \tag{3.2}$$

$$\begin{cases} z^{k+1} = 2Y_2 - (2Y_1 - z^k) \\ Y_1 = z^k + \frac{1}{4}hSY_1 \\ Y_2 = 2Y_1 - z^k + \frac{1}{4}hSY_2 \end{cases} \tag{3.3}$$

$$\begin{cases} z^{k+1} = 2Y_3 - (2Y_2 - 2Y_1 + z^k) \\ Y_1 = z^k + \frac{1}{2}ahSY_1 \\ Y_2 = 2Y_1 - z^k + \frac{1}{2}ahSY_2 \\ Y_3 = 2Y_2 - (2Y_1 - z^k) + (\frac{1}{2} - a)hSY_3 \end{cases} \tag{3.4}$$

$$\begin{cases} z^{k+1} = 2Y_4 - (2Y_3 - 2Y_2 + 2Y_1 - z^k) \\ Y_1 = z^k + \frac{1}{2}b_1hSY_1 \\ Y_2 = 2Y_1 - z^k + \frac{1}{2}b_2hSY_2 \\ Y_3 = 2Y_2 - (2Y_1 - z^k) + \frac{1}{2}b_3hSY_3 \\ Y_4 = 2Y_3 - (2Y_2 - 2Y_1 + z^k) + \frac{1}{2}b_4hSY_4 \end{cases} \tag{3.5}$$

where $a = 1.351207$ and $b_1 = -2.70309412$, $b_2 = -0.53652708$, $b_3 = 2.37893931$, $b_4 = 1.860681885$.

Let

$$S = \begin{bmatrix} 0 & M_1 \\ I & 0 \end{bmatrix}, \quad \text{or} \quad S = \begin{bmatrix} 0 & M_3 \\ M_3 & 0 \end{bmatrix}$$

(3.2)–(3.5) are,respectively, of order $o(\Delta t + \Delta x^2)$, $o(\Delta t^2 + \Delta x^2)$, $o(\Delta t^3 + \Delta x^2)$, $o(\Delta t^4 + \Delta x^2)$.

Let

$$S = \begin{bmatrix} 0 & M_2 \\ I & 0 \end{bmatrix}, \quad \text{or} \quad S = \begin{bmatrix} 0 & M_4 \\ M_4 & 0 \end{bmatrix}$$

(3.2)–(3.5) are, respectively, of order $o(\Delta t + \Delta x^4)$, $o(\Delta t^2 + \Delta x^4)$, $o(\Delta t^3 + \Delta x^4)$, $o(\Delta t^4 + \Delta x^4)$.

§4. NUMERICAL TEST

Consider equation

$$\begin{aligned} \frac{\partial V}{\partial t} &= \frac{\partial U}{\partial x} \\ \frac{\partial U}{\partial t} &= \frac{\partial V}{\partial x} \end{aligned} \tag{4.1}$$

with initial condition

$$V(x,0) = \cos \pi x \qquad U(x,0) = \cos \pi x. \tag{4.2}$$

This problem has an exact solution

$$V(x,t) = \cos \pi(x+t) \qquad U(x,t) = \cos \pi(x+t). \tag{4.3}$$

We apply schemes KTW(1.4), KTW(2.4), KTW(3.4), KTW(4.4) to this problem.

Let $\Delta x = \frac{2}{40}$. $V(m,n)$ denotes the value of $V(t,x)$ at $x = m\Delta x$, $t = n\Delta t$. V_m^n denote the approximating value of $V(m,n)$.

Numerical tests of stability in the critical case are given in the following tables. Scheme KTW(1,4) is not stable if $\frac{\Delta t}{\Delta x} = 1.4685$, but stable if $\frac{\Delta t}{\Delta x} = 1.4575$. Scheme KTW(2.4) is stable when $\frac{\Delta t}{\Delta x} = 1.4575$ but not stable when $\frac{\Delta t}{\Delta x} = 1.4675$. Scheme KTW(3,4) is not stable if $\frac{\Delta t}{\Delta x} = 1.8373$, but stable if $\frac{\Delta t}{\Delta x} = 1.8273$. Scheme KTW(4.4) is stable when $\frac{\Delta t}{\Delta x} = 1.1463$ but not stable when $\frac{\Delta t}{\Delta x} = 1.1563$. It can be easily seen that theoritical stability analysis comforms quite well with the numerical results.

Numerical tests of other schemes at the critical point are also in accord with theoritical analysis.

Scheme KTW(1.4)

Courant number	V_{16}^{99}	$V(16,99)$	error
$\frac{\Delta t}{\Delta x} = 1.4575$	1.059738	0.998936	−0.060801
$\frac{\Delta t}{\Delta x} = 1.4685$	−3.575937	0.976510	4.552447

Scheme KTW(2.4)

Courant number	V_{16}^{99}	$V(16,99)$	error
$\frac{\Delta t}{\Delta x} = 1.4575$	0.998016	0.998936	0.000920
$\frac{\Delta t}{\Delta x} = 1.4675$	47.828476	0.979739	−46.848724

Scheme KTW(3.4)

Courant number	V_{19}^{99}	$V(19,99)$	error
$\frac{\Delta t}{\Delta x} = 1.8273$	0.999751	0.999893	0.000141
$\frac{\Delta t}{\Delta x} = 1.8373$	0.335703	0.987792	0.652088
$\frac{\Delta t}{\Delta x} = 1.8473$	* * *	0.951742	* * *

* represents overflow

Scheme KTW(4.4)

Courant number	V_{7}^{99}	$V(7,99)$	error
$\frac{\Delta t}{\Delta x} = 1.1463$	0.997113	0.997048	0.000064
$\frac{\Delta t}{\Delta x} = 1.1563$	* * *	0.973123	* * *

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